Spectral theory for Schrödinger operators with singular interactions supported on curves and surfaces

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Insubria Summer School in Mathematical Physics
Spectral and scattering theory: From boundary value problems to selfadjoint operators

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This type of singular perturbations play an important role in models in mathematical physics (e.g. leaky quantum graphs) and are challenging from a **mathematical** perspective.

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Problems/questions that arise in a mathematical treatement are...

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The techniques in the (main) abstract parts of this lecture series apply to various spectral and boundary value problems, among them first order systems, Dirac operators, higher order ordinary and partial differential equations, infinite matrices...

Symmetric and selfadjoint operators

Definition (Adjoint operator)

Assume $\operatorname{dom} S$ is dense in $\mathcal{H}$. Then S^* is defined as

$$S^* g := g'$$
$$\operatorname{dom} S^* := \{g : \exists\, g' \text{ such that } (Sf, g) = (f, g') \text{ for all } f \in \operatorname{dom} S\}$$

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Let S be a symmetric operator in $\mathcal{H}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then

$$\|(S - \lambda)^{-1} g\| \leq \frac{1}{|\text{Im}\lambda|} \|g\|, \qquad g \in \text{ran}\,(S - \lambda).$$

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Definition (Decomposition of $\sigma(S)$)

- $\lambda \in \sigma_p(S) :\Leftrightarrow \ker(S-\lambda) \neq \{0\}$
- $\lambda \in \sigma_c(S) :\Leftrightarrow \ker(S-\lambda) = \{0\}, \overline{\operatorname{ran}(S-\lambda)} = \mathcal{H},$
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Proposition

Let S be a closed symmetric operator. Then

$$\mathbb{C} \setminus \mathbb{R} \subset (\sigma_r(S) \cup \rho(S)) \quad \text{and} \quad (\sigma_p(S) \cup \sigma_c(S)) \subset \mathbb{R}.$$

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In particular, $\sigma(S) = \sigma_p(S) \cup \sigma_c(S)$.

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Let A be an operator such that $\rho(A) \neq \emptyset$ and let $A \subset T$ with some operator T. Then

$$\text{dom}\,T = \text{dom}\,A \dot{+} \ker(T-\lambda), \qquad \lambda \in \rho(A).$$

Selfadjoint extensions via von Neumann's formula

von Neumann's first formula

Let S be a closed symmetric operator. Then

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Let S be a closed symmetric operator. Then S admits selfadjoint extensions in $\mathcal{H}$ if and only if

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In fact, the operator

$$A_U f = Sf_S + if_i - iUf_i, \quad f_S \in \mathrm{dom}\, S,\ f_i \in \ker(S^* - i)$$

is selfadjoint in $\mathcal{H}$ if and only if

$$U : \ker(S^* - i) \to \ker(S^* + i) \text{ is unitary.}$$

(Quasi) boundary triples, γ-fields and Weyl functions

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Examples: Sturm-Liouville operator in $L^2(0,1)$ and $L^2(0,\infty)$

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Example: $-\Delta + V$ in $L^2(\Omega)$ leads to $\{L^2(\partial\Omega), \gamma_D, -\gamma_N\}$

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Note that

$$\text{OBT} \quad \subset \quad \text{GBT} \quad \subset \quad \text{QBT}$$

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- $\Gamma = \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} S^* \to \mathcal{G} \times \mathcal{G}$ dense range
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Then

$$S := T \upharpoonright \big(\ker \Gamma_0 \cap \ker \Gamma_1\big)$$

is a densely defined operator in $\mathcal{H}$

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- $\ker \Gamma_0 \cap \ker \Gamma_1$ dense in $\mathcal{H}$

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$$S := T \upharpoonright \big(\ker \Gamma_0 \cap \ker \Gamma_1\big)$$

is a densely defined operator in $\mathcal{H}$ such that

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How to find QBT and T

Theorem

Assume that T is an operator in $\mathcal{H}$, $\mathcal{G}$ is a Hilbert space, and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ such that

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and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ QBT for S^* such that $A_0 = T \upharpoonright \ker \Gamma_0$.

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γ-field and Weyl function

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If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ OBT replace $\ker (T - \lambda)$ by $\ker (S^* - \lambda)$.

Example: $\gamma(\lambda)$ solves BVP, $M(\lambda)$ DN-map or ND-map

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We would like to find additional conditions on B or other quantities such that

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A Krein type resolvent formula

Selfadjoint extensions

Let $B = B^* \in \mathcal{L}(\mathcal{G})$ and consider the symmetric operator

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Example: $\{L^2(\partial\Omega), \gamma_N, \gamma_D\}$ for dom $T = H^2(\Omega)$, $B \in C^1(\partial\Omega)$ real

Schrödinger operators with δ-interactions

A suitable quasi boundary triple

Consider the operators

$$S = -\Delta + V,$$
$$\mathrm{dom}\, S = \{f \in H^2(\mathbb{R}^n) : f\restriction_\Sigma = 0\},$$

and

$$T = -\Delta + V,$$
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Proposition

Then $\overline{T} = S^*$ and $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$, where for $f = \binom{f_i}{f_e} \in \operatorname{dom} T$

$$\Gamma_0 f := \partial_{\nu_i} f_i\restriction_\Sigma + \partial_{\nu_e} f_e\restriction_\Sigma \quad \text{and} \quad \Gamma_1 f := f_i\restriction_\Sigma = f_e\restriction_\Sigma,$$

is a QBT such that $\operatorname{ran} \Gamma_0 = H^{1/2}(\Sigma)$ and $\operatorname{ran} \Gamma_1 = H^{3/2}(\Sigma)$, and

$$A_0 = T \restriction \ker \Gamma_0 = A_{free} \quad \text{and} \quad A_1 = T \restriction \ker \Gamma_1 = \begin{pmatrix} A_D^i & 0 \\ 0 & A_D^e \end{pmatrix}.$$

The corresponding γ-field and Weyl function

γ-field of the QBT $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$

For $\varphi \in H^{1/2}(\Sigma)$ and $\lambda \in \rho(A_{free})$ the function $\gamma(\lambda)\varphi$ solves the transmission problem

$$(-\Delta + V)u = \lambda u, \quad u_i\restriction_\Sigma = u_e\restriction_\Sigma, \quad \partial_{\nu_i} u_i\restriction_\Sigma + \partial_{\nu_e} u_e\restriction_\Sigma = \varphi.$$

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In fact, if $((A_{free} - \lambda)^{-1} f)(x) = \int_{\mathbb{R}^n} G_\lambda(x, y) f(y) dy$ then

$$(\gamma(\lambda)\varphi)(x) = \int_\Sigma G_\lambda(x, y)\varphi(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \Sigma.$$

The corresponding γ-field and Weyl function

Weyl function of the QBT $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has $M(\lambda) : L^2(\Sigma) \to L^2(\Sigma)$

$$M(\lambda)\varphi = \bigl(\Lambda_i(\lambda) + \Lambda_e(\lambda)\bigr)^{-1}\varphi, \quad \varphi \in \operatorname{dom} M(\lambda) = H^{1/2}(\Sigma),$$

where $\Lambda_i(\lambda)$ and $\Lambda_e(\lambda)$ denote Dirichlet-to-Neumann maps of the interior and exterior problem, respectively.

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Theorem

Let $\alpha \in W^{1,\infty}(\Sigma)$ be real. Then $A_{\delta,\alpha} = T \upharpoonright \ker(\Gamma_0 - \alpha\Gamma_1)$, i.e.,

$$A_{\delta,\alpha} = -\Delta + V,$$

$$\operatorname{dom} A_{\delta,\alpha} = \left\{ \begin{pmatrix} f_i \\ f_e \end{pmatrix} \in H^2(\Omega_i) \times H^2(\Omega_e) : \begin{array}{c} f_i\upharpoonright_\Sigma = f_e\upharpoonright_\Sigma, \\ \partial_{\nu_i} f_i\upharpoonright_\Sigma + \partial_{\nu_e} f_e\upharpoonright_\Sigma = \alpha f_i\upharpoonright_\Sigma \end{array} \right\}$$

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is selfadjoint in $L^2(\mathbb{R}^n)$ and for $\lambda \in \rho(A_{free})$:

$$\lambda \in \sigma_p(A_{\delta,\alpha}) \quad \Leftrightarrow \quad 0 \in \sigma_p(1 - \alpha M(\lambda)).$$

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Moreover, for $\lambda \in \rho(A_{free}) \cap \rho(A_{\delta,\alpha})$ one has

$$(A_{\delta,\alpha} - \lambda)^{-1} = (A_{free} - \lambda)^{-1} + \gamma(\lambda)\big(1 - \alpha M(\lambda)\big)^{-1} \alpha \gamma(\bar{\lambda})^*.$$

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Note $(A_{\delta,\alpha} - \lambda)^{-1} - (A_{free} - \lambda)^{-1}$ compact, in fact, $\mathfrak{S}_p$, $p > \frac{n-1}{3}$.

Approximation by regular potentials

Ω_ε ε-neighborhood of Σ, $W \in L^\infty(\mathbb{R}^n)$ with supp $W \subset \Omega_\beta$.

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$$W_\varepsilon(x) := \begin{cases} \frac{\beta}{\varepsilon} W\big(x_\Sigma + \frac{\beta}{\varepsilon} t\nu(x_\Sigma)\big), & x_\Sigma + t\nu(x_\Sigma) \in \Omega_\varepsilon, \\ 0, & \text{otherwise}, \end{cases}$$

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Theorem

The operators $H_\varepsilon = -\Delta + V + W_\varepsilon$ defined on $H^2(\mathbb{R}^n)$ converge in norm resolvent sense to $A_{\delta,\alpha}$, that is,

$$\lim_{\varepsilon \to 0+} \left\| (H_\varepsilon - \lambda)^{-1} - (A_{\delta,\alpha} - \lambda)^{-1} \right\| = 0.$$

Boundary triple techniques in spectral theory of PDEs – some references –

Main objective of the following reference list

In the following few slides some contributions in the area of boundary triples and extension theory for PDE's are listed. The following collection has an emphasis on quasi boundary triple techniques. It is by no means complete or an overview of the literature.

Since it is not easy to access the core ideas and complete proofs we highlight a small selection of papers and monographs which we recommend for a first reading; for this purpose we shall use the symbol

$$\heartsuit$$

Origins of (ordinary) boundary triples

- V. M. Bruk
  A certain class of boundary value problems with a spectral parameter in the boundary condition,
  Mat. Sb. (N.S.) **100 (142)** (1976), 210–216
- A. N. Kochubei
  Extensions of symmetric operators and symmetric binary relations [Russian], *Mat. Zametki* **17** (1975), 41–48;
  English translation: *Math. Notes* **17** (1975), 25–28

A further early reference is the monograph

- V. I. Gorbachuk and M. L. Gorbachuk
  Boundary Value Problems for Operator Differential Equations, Kluwer Academic Publ., Dordrecht, 1991

Pay also attention to the more general paper

- J. W. Calkin,
  Abstract symmetric boundary conditions,
  Trans. Amer. Math. Soc. **45** (1939), 369–442

Weyl function for ordinary boundary triples

- V. A. Derkach and M. M. Malamud
  Generalized resolvents and the boundary value problems for Hermitian operators with gaps,
  J. Funct. Anal. **95** (1991), 1–95
- V. A. Derkach and M. M. Malamud
  The extension theory of Hermitian operators and the moment problem, *J. Math. Sci.* **73** (1995), 141–242

Best available review paper and monograph touching these topics are

- ♡ J. Brüning, V. Geyler, and K. Pankrashkin
  Spectra of self-adjoint extensions and applications to solvable Schrödinger operators,
  Rev. Math. Phys. **20** (2008), 1–70
- ♡ K. Schmüdgen
  Unbounded Self-Adjoint Operators on Hilbert Space,
  Springer, Dordrecht, 2012

Quasi boundary triples and Weyl functions

- J. Behrndt and M. Langer
  Boundary value problems for elliptic partial differential operators on bounded domains,
  J. Funct. Anal. **243** (2007), 536–565
- ♡ J. Behrndt and M. Langer
  Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples, in: *Operator Methods for Boundary Value Problems*, London Math. Soc. Lecture Note Series, vol. 404, 2012, pp. 121–160

The papers can be downloaded from

`www.math.tugraz.at/~behrndt/`

QBTs for Schrödinger operators with δ-potentials

- J. Behrndt, M. Langer, and V. Lotoreichik
  Schrödinger operators with δ and δ'-potentials supported on hypersurfaces, *Ann. Henri Poincaré* **14** (2013), 385–423
- J. Behrndt, M. Langer, and V. Lotoreichik
  Spectral estimates for resolvent differences of self-adjoint elliptic operators,
  Integral Equations Operator Theory **77** (2013), 1–37
- J. Behrndt, G. Grubb, M. Langer, and V. Lotoreichik
  Spectral asymptotics for resolvent differences of elliptic operators with δ and δ'-interactions on hypersurfaces,
  J. Spectral Theory **5** (2015), 697–729
- J. Behrndt, R.L. Frank, C. Kühn, V. Lotoreichik, and J. Rohleder
  Spectral theory for Schrödinger operators with delta-interactions supported on curves in $\mathbb{R}^3$,
  Ann. Henri Poincaré **18** (2017), 1305–1347

QBT methods and Titchmarsh-Weyl theory for PDEs

- J. Behrndt and J. Rohleder
  Spectral analysis of selfadjoint elliptic differential operators, Dirichlet-to-Neumann maps, and abstract Weyl functions,
  Adv. Math. **285** (2015), 1301–1338
- J. Behrndt and J. Rohleder
  Titchmarsh-Weyl theory for Schrödinger operators on unbounded domains,
  J. Spectral Theory **6** (2016), 67–87
- J. Behrndt, M. Langer, V. Lotoreichik, and J. Rohleder
  Quasi boundary triples and semibounded self-adjoint extensions,
  Proc. Roy. Soc. Edinburgh Sect. A, to appear

QBTs for scattering problems and the SSF

- J. Behrndt, M.M. Malamud, and H. Neidhardt
  Scattering matrices and Weyl functions,
  Proc. London Math. Soc. **97** (2008), 568–598
- J. Behrndt, M.M. Malamud, and H. Neidhardt
  Scattering matrices and Dirichlet-to-Neumann maps,
  J. Funct. Anal. **273** (2017), 1970–2025
- J. Behrndt, F. Gesztesy, and S. Nakamura
  Spectral shift functions and Dirichlet-to-Neumann maps,
  Math. Ann., to appear

Ordinary boundary triples for PDEs

- G. Grubb
  A characterization of the non-local boundary value problems associated with an elliptic operator,
  Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 425–513
- B. M. Brown, G. Grubb and I. G. Wood
  M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems,
  Math. Nachr. **282** (2009), 314–347
- M. M. Malamud
  Spectral theory of elliptic operators in exterior domains,
  Russ. J. Math. Phys. **17** (2010), 96–125
- J. Behrndt and M. Langer
  Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples, in: *Operator Methods for Boundary Value Problems*, London Math. Soc. Lecture Note Series, vol. 404, 2012, pp. 121–160

Other related abstract boundary triple techniques

Generalized boundary triplets were studied in

- V. A. Derkach and M. M. Malamud
  The extension theory of Hermitian operators and the moment problem, *J. Math. Sci.* **73** (1995), 141–242
- V. A. Derkach, S. Hassi, M. M. Malamud, and H. de Snoo
  Boundary triplets and Weyl functions. Recent developments. in: *Operator Methods for Boundary Value Problems*, London Math. Soc. Lecture Note Series, vol. 404, 2012, pp. 161–220

and for boundary relations see

- V. A. Derkach, S. Hassi, M. M. Malamud, and H. de Snoo
  Boundary relations and their Weyl families, *Trans. Amer. Math. Soc.* **358** (2006), 5351–5400
- V. A. Derkach, S. Hassi, M. M. Malamud, and H. de Snoo
  Boundary relations and generalized resolvents of symmetric operators, *Russ. J. Math. Phys.* **16** (2009), 17–60

Other closely related techniques

- F. Gesztesy and M. Mitrea
  A description of all self-adjoint extensions of the Laplacian and Kreĭn-type resolvent formulas on non-smooth domains,
  J. Anal. Math. **113** (2011), 53–172
- A. Posilicano
  Boundary triples and Weyl functions for singular perturbations of self-adjoint operators,
  Methods Funct. Anal. Topology **10** (2004), 57–63
- A. Posilicano
  Self-adjoint extensions of restrictions,
  Operators and Matrices **2** (2008), 1–24
- A. Posilicano and L. Raimondi
  Kreĭn's resolvent formula for self-adjoint extensions of symmetric second-order elliptic differential operators,
  J. Phys. A: Math. Theor. **42** (2009), 015204, 11pp

Other closely related techniques

- ♡ O. Post
  Boundary pairs associated with quadratic forms,
  Math. Nachr. **289** (2016), 1052–1099
- V. Ryzhov
  A general boundary value problem and its Weyl function,
  Opuscula Math. **27** (2007), 305–331
- V. Ryzhov
  Weyl-Titchmarsh function of an abstract boundary value problem, operator colligations, and linear systems with boundary control,
  Complex Anal. Oper. Theory **3** (2009), 289–322

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